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STUDENTS' THINKING PROCESS IN INVESTIGATING MATHEMATICAL STATEMENT

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Abstract

The statement is a declaratif sentence that can be true or false. It can't do both at the same time so it needs an investigation through proving process. This research aims to explore the students' thinking process in investigating the truth value of matematical statement. This research is conducted to seventeen students of mathematical education whom are taking the Abstract Algebra Course. The subject selecting in this reasearch is based on the students' ability in doing the investigating by using formal proof. The data collecting uses the written test and the interview. Meanwhile, the data analysis is conducted through three steps: data reduction, data interpretation, and taking a conclusion. Based on the result of the research, it is found that students do the thinking process in investigating the matematical statements' truth value through four steps. The first step, students understand the statements by reading them and classifying them to be a number of objects. The second one, students determine the proof startegies based on the definition or axiom. The third one, students do the algebraic operation by using symbol manipulation. The last one, students provide justification as the form of their belief of the proof results.

Keywords: Investigating, Mathematical Statement, Thinking,

Abstrak

Pernyataan merupakan kalimat deklaratif yang dapat bernilai benar atau salah tetapi tidak bisa keduanya, sehingga perlu dilakukan penyelidikan melalui pembuktian. Penelitian ini bertujuan untuk mengeksplorasi proses berpikir mahasiswa dalam menyelidiki kebenaran pernyataan matematis. Penelitian ini dilakukan kepada tujuh belas mahasiswa program studi pendidikan matematika yang sedang menempuh mata kuliah aljabar abstrak. Pemilihan subjek pada penelitian ini didasarkan pada kemampuan mahasiswa dalam memberikan hasil investigasinya melalui pembuktian secara formal. Pengumpulan data pada penelitian ini menggunakan tes tertulis dan wawancara. Sedangkan analisis data dilakukan melalui reduksi data, interpretasi data, dan kesimpulan, Berdasarkan hasil penelitian ditemukan bahwa mahasiswa melakukan proses berpikir dalam menyelidiki kebenaran pernyataan matematis melalui empat tahapan. Tahap pertama, mahasiswa memahami pernyataan dengan cara membaca kemudian mengklasifikasikan menjadi beberapa objek. Tahap kedua, mahasiswa menentukan strategi pembuktian berdasarkan definisi atau aksioma. Tahap ketiga, mahasiswa melakukan operasi aljabar melalui manipulasi simbol. Tahap terakhir, mahasiswa memberikan justifikasi sebagai bentuk keyakinannya terhadap hasil pembuktian

***Kata kunci**: Berpikir, Menyelidiki, Pernyataan Matematika*

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INTRODUCTION

Mathematical logic contains objects in the form of statements and argument structures represented in formal proof (Engeler, 2018). The statement or proposition in mathematics is interpreted as declaratif sentence that can be true or false, but it can not do both at the same time (Imamoglu & Togrol, 2015). Thus, the truth of statement need to be investigated by thinking logically. The result of investigation can be mathematical arguments (Ogan-Bekiroglu & Eskin, 2012). An argument in mathematics can be expressed through both of formal and informal logics (Umah et al., 2016). Informal logic emphasizes on expressed argument that not contain components proportionally (Aberdein, 2019). Meanwhile, formal logic contains the set of premise that will establish a conclusion based on logic rules or deductive reasoning (Aristidou, 2020). Basically, the main objectives of proof are explaining, communicating, and estimating the statements to deductive system (Mukuka & Shumba, 2016).

A mathematical proof is formed through the thinking process that is sonducted by students. Thinking is a form of two ways communication between an individu and himself that no one else knows about it as it occurs in the brain. Thinking can be interpreted as the mental activity that happened in the brain to remember, understand, find or create a solution, analyze, synthesize a problem and solve it (Netti et al., 2016). In mathematical education context, thinking is students' mental structure that happened as long as they learn or solve the problem. This mental structure can be built through set-before and met-before. Set-before can be happened when students have the ability to interact and describe the important points about mathematics by using symbols, meanwhile met-before can be happened through previous experience to solve a new problem (Tall & Witzke, 2020).

Tall (2013) states that mathematical thinking is designed to give positive encouragement to students so they can be confident in completing the mathematical proof. Students do the thinking process through the categories of three world of mathematics theory. The first category, conceptual-embodied world that can be reflecting by observing, describing, defining, and concluding based on the experiment. The second one, the proceptual-symbolic world that can be reflecting by symbolic operation based on the mathematical concepts. The last one, the formal-axiomatic world defines the mathematical concepts as the axiom structure in formal proof (Tall & Witzke, 2020).

In the higher education, students need to have mathematical thinking skill because they are often faced with proving problem. This is because the proving problem is a tool to construct student's knowledge through thinking rationally as a decision-making process (Metaxas et al., 2016). Students at the university level must also have several skills, including: verifying contradictions, determining symbols, translating verbal into algebraic forms, routinely performing algebraic manipulation, and making arguments skills (Öztürk & Kaplan, 2019).

Furthermore, Bleiler et al., (2014) states that students do thinking process to indetify the components in mathematical problem, evaluate the alternative problem solving, determine the definition or axiom, and making the conclusion. The result of the research shows that mathematics teacher can do

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the mathematical thinking through six aspects: generalization, induction, using the symbols, logical thinking, and mathematical proof (Lopez & Tancinco, 2016). Meanwhile, the result of research conducted by As'ari et al., (2019) shows that the mathematical learning in school does not emphasize in students' thinking process but the corectness of the students' written answer test. However, students in college with high mathematical ability do the looking back the answer step and students with low mathematical ability do not that (Wardhani et al., 2016). Students construct the conjecture through cognitive process to solve the mathematical problem related to the analysis (Astawa et al., 2018). Students can do the analytically thinking process to produce the mathematical argument (Khusna, 2020). Furthermore, another result of the research shows that logical thinking is needed in algebraic proof as the form of deductive reasoning (Ramirez-Ucles & Ruiz-hidalgo, 2022).

Based on the previous research, it is showed that students can think logically to do the algebraic proof. However, the result of the observation shows that not all students do formal proof. This students' inability can be caused they do noy understand the concept that will be used or they do not be careful when understanding the statement that will be proven. Therefore, it important to do a research that aims to explore the students' thinking process in investigating the truth value of mathematical statement.

METHOD

This qualitative research is conducted to 17 students of mathematics education study program at Universitas Hasyim Asy'ari whom studying abstract algebra. The researcher selects the subjects by using purposive sampling technique, researcher choose the students whom be able to investigate the mathematical statement through formal proof by using the definition or axiom. The students that complete the test without using the definition or axiom can not be selected as the subject of the interview. Furthermore, the ability to communicate verbally is one of the consideration in selecting the subject of the research.

The research is conducted by passing a number of steps. The first step is doing an observation as long as the abstract algebra learning is going on to determine the students' verbally mathematical ability. The second one is giving the test to get the data about the students' thinking process in investigating the trust value of mathematical statement. This test is also used to get the data about students' understanding about first two conditions of group: fulfilling closed and assosiative properties. If students do the error in first two conditions, it can be confirmed that they also will do the error in the last two conditions (about the identity and inverse elements). The third one is observing the resut of the test then conducting the interview to the selected subjects. The interview process is recorded by using the tape recorder so it can simplify the transcription process. The test can be shown as following:

"For each statements below, please check whether it true or false.

1. $G = \{-1,1\}$ with addition operation is closed.
2. The set Z with binary operation $*$ which $a * b = a + 2b$ with a and b are integers fulfills the associative property"

The data analysis is conducted through three steps: data reduction, data interpretation, and making a coclusion.

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Data reduction is choosing and focusing the appropriate data with the research abjectives. The inappropriate data can be considering as the findings. Meanwhile, data interpretation is describing the reduced data so the conclusion can be easier to be arranged.

In this research, the students' thinking processes are adopted from Öztürk & Kaplan (2019) and Kidron & Tall (2015) as shown in Table 1. Öztürk & Kaplan (2019) conducted a research on cognitive and metacognitive processes of mathematics teachers. Meanwhile, Kidron & Tall (2015) examined the algebraic proof through symbol manipulation.

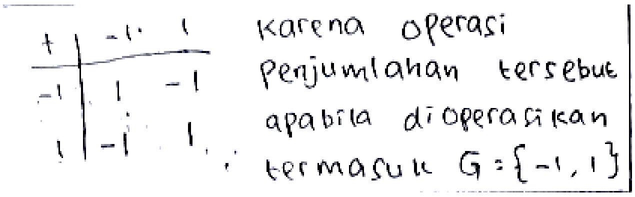
Table 1. The steps of thinking process

Steps	Description
Reading the statements	Identify the objects of the statements
Determining the strategy	Determine the mathematical rules to be used.
Manipulation the symbols	Elaborate the proof through algebraic operation
Giving the justification	Re-check the result of the proof and make a conclusion
	Being confidence about produced conclusion

RESULT AND DISCUSSION

The result of the test shows that eleven students did the investigation through formal proof. Meanwhile, five students did not do the formal proof as they just wrote "true" of "false" at their answer sheets. Therefore, those five students can not be considered as the subject of the interview. Then, the subject of the interview is chosen randomly from eleven possible students.

Based on the result of the research, it is known if the subject did the investigation about the test number 1 by: (1) reading and observing every given information then classifying them to be several objects. (2) determining the used stategy to prove. This stategy can be embodied by using Cayley table to prove the closure of addition property. Meanwhile, subject did the error with the calculation as he did the multiplication over addition. (3) giving the justification that $G = \{-1,1\}$ with the addition operation is closed as the calculation result is also the elements of G. Subject did not realize that he did the error as he did the multiplication over addition on Cayley table. Based the result of the investigation, it is known that subject did not do the symbol manipulation and is not precise when investigating the test number 1. The investigation result about the test number 1 can be shown below in Figure 1.



Translation:
As the addition if operated, it is also $G = \{-1,1\}$

Figure 1. Subjects' Answer of Number 1

Meanwhile, the investigation result about the test number 2 shows that the first step he done was reading the given information and classifying

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them into several objects: (a) $a * b = a + 2b$ for all a and b are integers; (b) binary operation; and (c) associative properties. The second step is determining the startegy about associative property in definition of group. Subject did the algebraic operation through symbol manipulation to check the truth of the statement. Subject did the algebraic operation by using the analytically thinking skill to elaborate $a * (b * c) = a * (b + 2c)$ and $(a * b) * c = (a + 2b) * c$. Subject supposed a as b and $b + 2c$ as c to obtain $b + 2c = a + 2(b + 2c) = a + 2b + 4c$. Subject also supposed $a + 2b$ as a and c as b to obtain $a + 2b = (a + 2b) + 2c = a + 2b + 2c$. Subject elaborated each symbols to prove the statement number 2.

The last step is giving the justification of the result proof. The justification is "a integer set with binary operation $*$ with $a * b = a + 2b$ does not fulfill the associative property because $a * (b * c) = a + 2b + 4c$ and $(a * b) * c = a + 2b + 2c$ so it can be concluded that $a * (b * c) \neq (a * b) * c$. The justification is a conclusion from the conducted proof. The investigation result about the test number 2 can be shown in Figure 2.

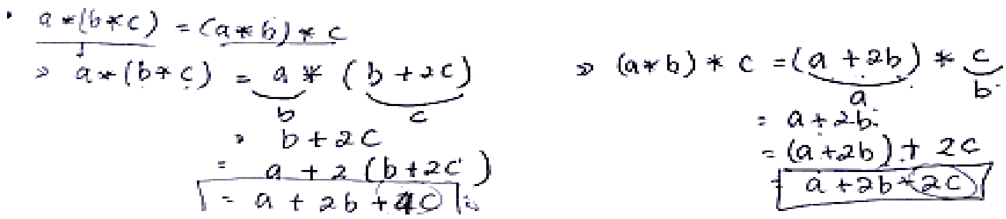

Figure 2. Subjects' Answer of Number 2

Based on the exposure above, it can be known that student investigates the mathematical statementthrough four steps: reading the statements, determine the strategy, manipulate the symbols, and giving justification. Those four steps can be describe on Table 2.

The result of the research shows that the research subject do the mathematical thinking process in investigating the abstract algebra statement through four steps. Those four steps are:

The first step is reading and classifying the given informations into several objects. This is in line with Öztürk & Kaplan (2019) that stated if reading is an initial step to prove the preposition. The mathematical preposition is interpreted as the statement whom can be true or false. The value of the preposition is called the truth value (Imamoglu & Togrol, 2015).

The second one is determining the strategy to prove the statement. Stategy used by student to investigate the truth of the statement is by using the first two condition of the group definition. The definition of groups contains four conditions: fulfill the (1) closed and (2) associative properties, (3) the set has an identity element, and (4) every element of the set has an inverse (Faizah et al., 2020). The abstract definition can be used as an approach in solving procedure to do manipulation (Sutini et al., 2020).

The third one is manipulating symbols based on the conducted classification.

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Symbol manipulation is used to elaborate the proof based on the analytically thinking skill (Faizah et al., 2020). thinking analytically is done by separating objects with provious objects into component part (Thaneerananon et al., 2016). In this case, student separates the proved object by elaborating the steps of proof through algebraic operation one by one.

Table 2.Students' Thinking Process

Step	Statement 1	Statement 2
1. Reading the statement	Reading the statement that will be proved, then classifying it into the followings: - The known non-empty set is $G = \{-1,1\}$, - Using addition operation - Closed properties	Classifying general information: - $a * b = a + 2b$ for all $a\ and\ b$ are integers - Binary operation - Associative properties
2. Determining strategy	A non-empty set of G = $\{-1,1\}$ is closed under addition operation	A non-empty set Z is associative under the binary operation * which $x * y = x + 2y$ for every $x, y, z \in G$ so $x * (y * z) = (x * y) * z$.
3. Symbol manipulation	Write the addition notation on the answer sheet but conduct the multiplication operation over addition.	- Elaborate $a * (b * c) = a * (b + 2c)$ by supposing a as b, and $b + 2c$ as c (to simplify the symbol manipulation). Then it obtained that $b + 2c = a + 2(b + 2c) = a + 2b + 4c$ - Elaborate $(a * b) * c = (a + 2b) * c$ by supposing $a + 2b$ as a and c as b. Then it be obtained that $(a + 2b) + 2c = a + 2b + 2c$
4. Justifying	- "$G = \{-1,1\}$ with closed to addition operation because the operation result on the Cayley table is an element of G." - Not realizing that the multiplication operation taken does not match what is written in the worksheet	- A set of integers with binary operation $a * b = a + 2b$ does not fulfill the associative property as $a * (b * c) \neq (a * b) * c$

The last step is giving the justification as the conlusion from the result proof. Student can be said do the justification if he make a decision and give the reason behind as the validation (Chua, 2018). This decision can be a conclusion of a formally formed argument in deductive proof. A

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conclusion can be said be true or valid if the reason is deductive logic or contadiction proof (Chua, 2016).

Öztürk & Kaplan (2019) stated that preposition proof in algebra can be performed through cognitive activity as follows: reading the preposition, evaluating the truth value, determining the strategy, making the plan, and thinking shortcuts or quick methods. However, the result of the research shows that not all of cognitive activities studied by Ozturk & Kaplan in proving the mathematical preposition is performed by student when investigating the truth of the statement. Student did not do the evaluation of proof result, so they do not realize that they complete the proof carelessly. In this case, student give the justification after doing algebraic operation through symbol manipuation. Thus, it needs to be given the scaffolding to students whom feel the difficulty when solving the abstract algbera problem so they can complete the proof carefully and formally.

CONCLUSION AND SUGGESTION

Students can investigate the truth of the mathematical statement related to abstract algebra throug thinking process by using four steps: reading the statement, determining the strategy, manipulating the symbols, and justification. Those four steps are performed by students when investigating the mathematical statement so the proof can be general. However, it is found that student just write "true" or "false" on their answer sheet, thus they can not be considered as the subject of the interview. This findings directs to the advanced research that study abouth the giving the scaffolding to students whom face the difficulty when performing the abstract algebra proof. This scaffolding works as the support to students so they can do the algebraic proof formally and generally.

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